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CONTINUOUS MULTISTATE STRUCTURE FUNCTIONS

by

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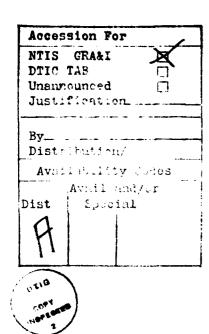Ninth Floor, Schenley Hall
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ABSTRACT

Multistate structure functions on state spaces which are not necessarily finite are discussed. Integral representations are derived which extend the decomposition of Block and Savits (1982). Bounds for the structure function are obtained and the representation is applied to the Barlow-Wu structure function and to nondecreasing homogeneous functions.

0. Introduction

Multistate structure functions have been discussed by Barlow and Wu (1978), El-Neweihi, Proschan and Sethuraman (1978), Griffith (1980) and Block and Savits (1982). All of these papers have limited their discussion to the case where the state space is finite.

In this paper we consider the situation where the structure function is defined on some subset of $\mathbf{R}^n$. Results similar to those obtained in the other papers, especially Block and Savits (1982) are derived here. The complication is that topological considerations must be considered.

In Section 1, we obtain an integral representation for a general multi-state structure function on $\mathbf{R}^n_+$. Properties of the upper sets associated with the function are discussed and the integrand in the representation is described in terms of the extreme points of its upper sets. This gives the extension of the min path representation from the finite case. Minimality of the representation is also demonstrated.

In Section 2, we consider the more difficult case of the function defined on some $\Delta \subset \mathbf{R}^n_+$. Results similar to those in Section 1 as well as bounds are obtained.

The decomposition obtained in Sections 1 and 2 are applied to systems of the type discussed by Barlow and Wu (1978). As a second application, a representation for nondecreasing homogeneous functions is obtained.

In general we limit our discussion to sets $A \subset \mathbf{R}^n$. By A^o, $\bar{A}$ and $B(A)$ we mean the topological interior, closure and boundary of A. For $\underline{x} = (x_1, \ldots, x_n)$, $\underline{y} = (y_1, \ldots, y_n)$ in $\mathbf{R}^n$, $\underline{x} \le \underline{y}$ means $x_i \le y_i$ for $i = 1, \ldots, n$, $\underline{x} < \underline{y}$ means $x_i < y_i$ for $i = 1, \ldots, n$, and $\underline{x} \lneq \underline{y}$ means $\underline{x} \le \underline{y}$ and $\underline{x} \ne \underline{y}$. We also define

$$L_Q(\underline{x}) = \{\underline{y}: \underline{y} \le \underline{x} \}, \quad L_Q^o(\underline{x}) = \{\underline{y}: \underline{y} < \underline{x} \},$$

$$U_Q(\underline{x}) = \{\underline{y}: \underline{y} \ge \underline{x} \}, \quad U_Q^o(\underline{x}) = \{\underline{y}: \underline{y} > \underline{x} \}.$$

Finally we say a function $\Phi: \Delta \rightarrow \mathbf{R}^n$ where Δ is a Borel measurable subset of $\mathbf{R}^n$ is a <u>multistate monotone structure function</u> (MMS) if Φ is Borel measurable and nondecreasing (i.e., $\underline{x} \le \underline{y}$ implies $\Phi(\underline{x}) \le \Phi(\underline{y})$). A set $A \subset \mathbf{R}^n$ is said to be an <u>upper</u> (<u>lower</u>) <u>set</u> if $\underline{x} \in A$ and $\underline{x} \le (\ge) \underline{y}$ implies $\underline{y} \in A$.

1. The $\mathbf{R}_+^n$ Case

For purposes of exposition we first consider the special case of a multistate monotone structure function $\Phi: [0,\infty)^n \rightarrow [0,\infty)$ which is assumed to be right-continuous i.e. for each $\underline{x} \in \mathbf{R}_+^n$ and for each $\varepsilon > 0$, there is a $\delta > 0$ such that for $\underline{x} \le \underline{y} \le \underline{x} + \delta\underline{1}$, $|f(\underline{x}) - f(\underline{y})| < \varepsilon$. Here we will obtain an integral representation for Φ in terms of binary valued functions each of which can be expressed via its "min path sets". The min cut representation leads to technical difficulties and hence will be deferred until Section 2.

1.1 Integral representation

Let $\Phi: [0,\infty)^n \rightarrow [0,\infty)$ be a right-continuous MMS. For each $t \ge 0$ we let

$$U_t = \{\underline{x}: \Phi(\underline{x}) \ge t\}.$$

(1.1) <u>Proposition</u>. Each U_t is a closed upper set and $U_t \subset U_s$ for $t \ge s \ge 0$.

<u>Proof</u>. We need only show that U_t is closed since the other results are obvious. Suppose that $\Phi(\underline{x}) < t$. It follows by right-continuity that there exists $\underline{z} > \underline{x}$ such that $\Phi(\underline{z}) < t$. Hence for every $0 \le \underline{y} < \underline{z}$ we have $\Phi(\underline{y}) < t$. Thus the complement of U_t is open (relative to $[0,\infty)^n$) and so U_t is closed.

We now define a binary valued function ϕ by

$$(1.2) \qquad \phi(\underline{x}, t) = I_{U_t}(\underline{x}) = I_{[0, \Phi(\underline{x})]}(t)$$

for all $t \geq 0$, $\underline{x} \geq \underline{0}$, where I_A is the indicator function of the set A. It is clear that for fixed t, ϕ is Borel measurable and nondecreasing in $\underline{x}$, while for fixed $\underline{x}$, ϕ is left-continuous and nonincreasing in t. Also $\Phi(\underline{x}) \geq t$ if and only if $\phi(\underline{x}, t) = 1$.

The next result follows easily and is analogous to the finite state decomposition given in Theorem 2.8 of Block and Savits (1982).

(1.3) **Theorem.**
$$\Phi(\underline{x}) = \int_0^\infty \phi(\underline{x}, t) dt.$$

In order to obtain a representation of ϕ in terms of its "min path sets" we need to first investigate the nature of upper sets in more detail.

1.2 Upper sets and extremal points

We first list some elementary facts about upper sets.

(1.4) **Proposition.** Let A, $\{A(t): t \in T\}$ be upper sets where T is some index set (i.e., $T \subset \mathbb{R}$).

(i) $\underset{t \in T}{\cup} A(t)$ and $\underset{t \in T}{\cap} A(t)$ are upper sets.

(ii) $A = \underset{\underline{x} \in A}{\cup} U_Q(\underline{x})$.

(iii) $\overline{A}$ and A^o are upper sets.

Proof. The first two statements are obvious. Suppose now that $\underline{x} \in \overline{A}$. Then there exist $\langle \underline{x}_n \rangle \subset A$ with $\underline{x}_n \to \underline{x}$. If $\underline{y} \geq \underline{x}$, we set $\underline{y}_n = \underline{x}_n + (\underline{y} - \underline{x})$. Clearly $\underline{y}_n \in A$ and $\underline{y}_n \to \underline{y}$; i.e., $\underline{y} \in \overline{A}$ and so $\overline{A}$ is an upper set. Suppose now that $\underline{x} \in A^o$. Then there exists some $\delta > 0$ such that $N_\delta(\underline{x}) = \{\underline{z}: |z_i - x_i| < \delta \text{ all } i\} \subset A$. It thus follows that if $\underline{y} \geq \underline{x}$, then $N_\delta(\underline{y}) \subset A$ because whenever $\underline{z} \in N_\delta(\underline{y})$, $\underline{z} \geq \underline{z} - (\underline{y} - \underline{x}) \in N_\delta(\underline{x})$. Hence $\underline{y} \in A^o$ and so A^o is an upper set.

Thus if A is any set, the intersection of all closed upper sets containing A is a closed upper set containing A, which we call the <u>closed upper hull</u> of A. It is the smallest closed upper set containing A.

We now introduce some further concepts analogous to the notion of extreme points for convex sets. Let A be any upper set. We call $\underline{x}$ a lower extreme point of A if $L_Q(\underline{x}) \cap \bar{A} = \{\underline{x}\}$ and we denote the set of all lower extreme points of A by $E_L(A)$. Note that $E_L(A)$ must be contained in the topological boundary of A.

Before we prove our main result we need the following lemma.

(1.5) <u>Lemma</u>. Let A be a closed upper set in $\mathbf{R}_+^n = [0,\infty)^n$ and let $E_L(A)$ be its set of lower extreme points. Then $\underline{x} \in A$ if and only if $\underline{x} \geq \underline{y}$ for some $\underline{y} \in E_L(A)$.

<u>Proof</u>. Clearly we only need prove that if $\underline{x} \in A$ then $\underline{x} \geq y$ for some $\underline{y} \in E_L(A)$. So let $\underline{x} = (x_1, \ldots, x_n) \in A$. Set $y_1 = \inf\{0 \leq x \leq x_1 : (x, x_2, \ldots, x_n) \in A\}$. Since A is closed it follows that $(y_1, x_2, \ldots, x_n) \in A$. Now set $y_2 = \inf\{0 \leq x \leq x_2 : (y_1, x, x_3, \ldots, x_n) \in A\}$. Again we have that $(y_1, y_2, x_3, \ldots, x_n) \in A$. Thus inductively we can find $\underline{y} = (y_1, \ldots, y_n) \leq \underline{x}$ such that $(y_1, \ldots, y_i, x_{i+1}, \ldots, x_n) \in A$ for all $i = 1, \ldots, n$. We now claim that $\underline{y} \in E_L(A)$. Suppose that $\underline{z} \in A \cap L_Q(\underline{y})$. Then $\underline{z} \in A$ and $\underline{z} \leq \underline{y} \leq \underline{x}$. If $\underline{z} \neq \underline{y}$, then they differ in at least one coordinate. Let the first such coordinate be i. Then $z_1 = y_1, \ldots, z_{i-1} = y_{i-1}, z_i < y_i \leq x_i$. This implies that $(y_1, \ldots, y_{i-1}, z_i, x_{i+1}, \ldots, x_n) \in A$. But this contradicts the definition of y_i and so $\underline{z} = \underline{y}$. Hence $\underline{y} \in E_L(A)$.

(1.6) <u>Theorem</u>. Let A be a closed upper set in $\mathbf{R}_+^n$. Then A is the closed upper hull of its lower extreme points. Furthermore, we have $A = \bigcup\limits_{\underline{y} \in E_L(A)} U_Q(\underline{y})$.

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Proof. Let B be the closed upper hull of $E_L(A)$ (i.e., $B = \cap\{U: U \text{ closed upper}$ set and $U \supset E_L(A)\}$). Since $A = \cup_{y \in A} U_Q(\underline{y})$ and A is closed, it follows that

$$A \supset \overline{\cup_{\underline{y} \in E_L(A)} U_Q(\underline{y})} \supset B \supset \cup_{\underline{y} \in E_L(A)} U_Q(\underline{y}).$$ But according to Lemma 1.5,

$$A \subset \cup_{\underline{y} \in E_L(A)} U_Q(\underline{y})$$ and we are done.

(1.7) **Corollary.** If A is a closed upper set in $\mathbb{R}^n_+$ and I_A is its indicator function, then

$$(1.8) \qquad\qquad I_A(\underline{x}) = \max_{\underline{y} \in E_L(A)} \; \min_{1 \leq i \leq n} \alpha_{i, y_i}(\underline{x})$$

where $\alpha_{i,t}(\underline{x}) = 1$ if $x_i \geq t$ and 0 otherwise.

Proof. Now $I_A(\underline{x}) = 1 \iff \underline{x} \in A \iff \underline{x} \geq \underline{y}$ for some $\underline{y} \in E_L(A) \iff \alpha_{i, y_i}(\underline{x}) = 1$ for all $i = 1, \ldots, n$, some $\underline{y} \in E_L(A)$.

(1.9) **Remarks.** (i) If we restrict $\underline{x}$ in (1.8) to $\mathbb{R}^n_+$, then we need only take the minimum over all i such that $y_i \neq 0$; (ii) the results (1.5), (1.6) and (1.7) remain valid for any closed upper set in $\mathbb{R}^n$ which is lower bounded i.e. there is a $\underline{y} \in \mathbb{R}^n$ such that $\underline{y} \leq \underline{x}$ for all $\underline{x}$ in the set; (iii) in case A is an open upper set in $\mathbb{R}^n$ which is lower bounded we have the analogous result that $A = \cup_{\underline{y} \in E_L(A)} U_Q^o(\underline{y})$.

This is true since if $\underline{x} \in A$ and A is open, there exists some $\underline{z} < \underline{x}$ with $\underline{z} \in A \subset \bar{A}$. Hence there exists $\underline{y} \in E_L(\bar{A}) = E_L(A)$ such that $\underline{y} \leq \underline{z}$ by Remark (ii) applied to Lemma 1.5. Consequently $\underline{x} \in U_Q^o(\underline{y})$. On the other hand, if $\underline{x} \in U_Q^o(\underline{y})$ for some $\underline{y} \in E_L(A)$, then $\underline{x} > \underline{y}$. But $\underline{y} \in \bar{A}$ and so there exists $\langle \underline{y}_n \rangle \subset A$ with $\underline{y}_n \to \underline{y}$. Thus eventually $\underline{x} > \underline{y}_n$ and so $\underline{x} \in A$.

We conclude this subsection by showing that the representation (1.8) is minimal.

(1.10) _Theorem_. Let A be a closed upper set in $\mathbb{R}_+^n$. Suppose we can write

$$I_A(\underline{x}) = \max_{\underline{y} \in E} \; \min_{1 \leq i \leq n} \alpha_{i,y_i}(\underline{x})$$

for some subset E of $\mathbb{R}_+^n$. Then $E \supset E_L(A)$.

Proof. Let $\underline{z} \in E_L(A)$. Since $\underline{z} \in A$, $I_A(\underline{z}) = 1$ and so there exists some $\underline{y} \in E$ such that $\alpha_{i,y_i}(\underline{z}) = 1$ for all $i = 1,\ldots,n$; i.e., $\underline{z} \geq \underline{y}$. Now if $\underline{z} \neq \underline{y}$, then $\underline{y} \notin A$ since $L_0(\underline{z}) \cap A = \{\underline{z}\}$. But then $I_A(\underline{y}) = 0$, which contradicts the above representation assumption. Hence $\underline{z} = \underline{y}$ and so $E \supset E_L(A)$.

1.3 Min path representation

We now combine the results of Sections 1.1 and 1.2. Let $\Phi:[0,\infty)^n \to [0,\infty)$ be a right-continuous MMS. Then according to Theorem 1.3 we have $\Phi(\underline{x}) = \int_0^\infty \phi(\underline{x},t)dt$ for all $\underline{x} \in \mathbb{R}_+^n$, where $\phi(\underline{x},t) = I_{U_t}(\underline{x})$. But from (1.8) and Remark (1.9i) we can write

$$(1.11) \qquad \phi(\underline{x},t) = \max_{\underline{y} \in P_t} \; \min_{i:y_i \neq 0} \alpha_{i,y_i}(\underline{x})$$

where $P_t = E_L(U_t)$. Note that $\underline{y} \in P_t$ if and only if $\Phi(\underline{y}) \geq t$ and $\Phi(\underline{z}) < t$ for all $\underline{z} \leq \underline{y}$, $\underline{z} \neq \underline{y}$. Thus we could call such $\underline{y}$ an upper critical vector for level t of Φ and $\{i: y_i \neq 0\}$ the corresponding min path set. Hence the result (1.11) is analogous to the min path representation in the finite state case.

2. Generalizations

In order to obtain a min cut representation for ϕ in (1.2) and also in order to be able to deduce the known results for the binary or finite state case we are forced to consider a more general setting. So let $\Phi: \Delta \to [0,\infty)$ be an MMS.

Before we consider the min cut representation, we must first rederive the results of Section 1 in this more general context. A subset $A \subset \Delta$ is said to be an upper set in (or relative to or with respect to) Δ if $\underline{x} \in A$ and $\underline{y} \in \Delta$ with $\underline{y} \geq \underline{x}$ implies that $\underline{y} \in A$. A subset $B \subset \Delta$ is said to be a lower set in Δ if $\Delta \backslash B$ is an upper set in Δ. Note that A is upper set in Δ if and only if $A = U \cap \Delta$ where U is an upper set (in $\mathbb{R}^n$). To see this simply set

$$U = \bigcup_{\underline{x} \in A} U_Q(\underline{x}).$$

2.1 Integral and min path representation

Let $\Phi: \Delta \to [0, \infty)$ be an MMS. As before we set $U_t = \{\underline{x} \in \Delta : \Phi(\underline{x}) \geq t\}$ for all $t \geq 0$. Then U_t is a Borel measurable upper set in Δ and $U_t \subset U_s$ if $t \geq s \geq 0$. If we set $\phi(\underline{x}, t) = I_{U_t}(\underline{x}) = I_{[0, \Phi(\underline{x})]}(t)$ for $t \geq 0$, $\underline{x} \in \Delta$, then ϕ has all the properties as in Section 1; i.e., for fixed t, ϕ is Borel measureable and nondecreasing in $\underline{x}$ on Δ and for fixed $\underline{x}$, ϕ is left-continuous and nonincreasing in $t \geq 0$. Clearly the following integral representation is still valid.

(2.1) <u>Theorem</u>. For $\underline{x} \in \Delta$, $\Phi(\underline{x}) = \int_0^\infty \phi(\underline{x}, t) dt$.

(2.2) <u>Remark</u>. It is sometimes convenient to consider the following alternative representation. Let $U_t^{(o)} = \{\underline{x} \in \Delta : \Phi(\underline{x}) > t\}$ and set $\xi(\underline{x}, t) = I_{U_t^{(o)}}(\underline{x}) = I_{[0, \Phi(\underline{x}))}(t)$ for $t \geq 0$, $\underline{x} \in \Delta$. Then $\Phi(\underline{x}) = \int_0^\infty \xi(\underline{x}, t) dt$. Note that for fixed $\underline{x}$, $\xi(\underline{x}, t)$ is now right-continuous in $t \geq 0$. Note also that $\xi \leq \phi$.

(2.3) <u>Proposition</u>. Let A and $\{A(t) : t \in T\}$ be upper sets in Δ.

    (i) $\bigcup_{t \in T} A(t)$ and $\bigcap_{t \in T} A(t)$ are upper sets in Δ.

    (ii) $A = \bigcup_{\underline{x} \in A} U_Q(\underline{x}) \cap \Delta$.

    (iii)   If Δ is a product set, then the closure of A in Δ and the interior of A in Δ are upper sets in Δ.

Proof. Again we need only prove (iii). Now the closure of A in Δ is $\bar{A} \cap \Delta$.

So suppose $\underline{x} \in \bar{A} \cap \Delta$ and $\underline{y} \geq \underline{x}$, $\underline{y} \in \Delta$. Then there exists $\langle \underline{x}_n \rangle \subset A$ with $\underline{x}_n \to \underline{x}$.

Set $\underline{y}_n = \underline{x}_n \vee \underline{y}$. Since Δ is a product set, $\underline{y}_n \in \Delta$ and so $\underline{y} \in \bar{A} \cap \Delta$ because $\underline{y}_n \to \underline{y}$.

Let B be the interior of A in Δ. Suppose $\underline{x} \in B$ and $\underline{y} \geq \underline{x}$ with $\underline{y} \in \Delta$. Then

there exists $\delta > 0$ such that $N_\delta(\underline{x}) \cap \Delta \subset A$ where $N_\delta(\underline{x}) = \{\underline{w}: |x_i - w_i| < \delta \text{ all } i\}$.

We will now show that $N_\delta(\underline{y}) \cap \Delta \subset A$ also and hence $\underline{y} \in B$. For let $\underline{z} \in N_\delta(\underline{y}) \cap \Delta$.

We define $w = (w_1, \ldots, w_n)$ by $w_i = z_i$ if $z_i < x_i$ and $w_i = x_i$ if $z_i \geq x_i$.

Since Δ is a product set, $\underline{w} \in \Delta$. Also $\underline{z} \geq \underline{w}$. Thus if we show that $\underline{w} \in N_\delta(\underline{x})$,

we are done. But $|x_i - w_i| = 0$ if $z_i \geq x_i$ and, if $z_i < x_i$, $|x_i - w_i| = |x_i - z_i| =$

$x_i - z_i \leq y_i - z_i < \delta$.

(2.4) Remark. It is not hard to show that (iii) fails if Δ is not a product set.

Now let A be an upper set in Δ. We say that $\underline{y} \in \Delta$ is a lower extreme

point of A relative to Δ if $L_Q(\underline{y}) \cap \bar{A} \cap \Delta = \{\underline{y}\}$. We denote the set of all lower

extreme points of A relative to Δ by $E_L^\Delta(A)$. Here $\bar{A}$ is the ordinary closure

of A and so $\bar{A} \cap \Delta$ is the closure of A in Δ.

In order to obtain the corresponding version of Lemma 1.5 we need that

Δ is complete in $\mathbb{R}^n$. Henceforth we will always assume that Δ is a closed

subset of $\mathbb{R}^n$.

(2.5) Lemma. Let A be a closed upper set in Δ which is lower bounded in

Δ (i.e., there is an $\underline{x} \in \Delta$ such that $\underline{x} \leq \underline{y}$ for all $\underline{y} \in A$). Then for $\underline{x} \in \Delta$, it

follows that $\underline{x} \in A$ if and only if $\underline{x} \geq \underline{y}$ for some $\underline{y} \in E_L^\Delta(A)$.

Proof. Again we need only prove the "only if" part. So let $\underline{x} = (x_1, \ldots, x_n) \in A$.

Define $\underline{w}^o = \underline{x}$ and set $A_1 = A \cap L_Q(\underline{w}^o)$. Its projection $B_1 = \pi_1 A_1$ where

$\pi_1(z_1, \ldots, z_n) = z_1$ is thus a nonempty subset in $\mathbb{R}^1$ which is bounded below.

Let $y_1 = \inf B_1$. Hence there exists $\underline{x}_m = (x_{1m}, \ldots, x_{nm}) \in A_1 \subset A$ such that

$x_{1m} \to y_1$ as $m \to \infty$. But the sequence $\langle \underline{x}_m \rangle$ is bounded in $\mathbb{R}^n$ and so has a convergent subsequence. Let $\underline{w}^1$ be its limit which lies in A_1 since A is closed. Note that $\underline{w}^1 \leq \underline{w}^0$ and $w_1^1 = y_1 \leq z_1$ for all $\underline{z} \in A_1$. Now set $A_2 = A \cap L_Q(\underline{w}^1)$ and let $B_2 = \pi_2 A_2$ be its projection where $\pi_2(z_1, \ldots, z_n) = z_2$. By the same argument, there exists $\underline{w}^2 \in A_2$ such that $w_2^2 = y_2 \leq z_2$ for all $\underline{z} = (z_1, z_2, \ldots, z_n) \in A_2$, where $y_2 = \inf B_2$. Continuing by induction we find that for each $i = 1, \ldots, n$ there exists $\underline{w}^i \in A_i = A \cap L_Q(\underline{w}^{i-1})$ such that $w_i^i = y_i \leq z_i$ for all $\underline{z} = (z_1, \ldots, z_n) \in A_i$ where $y_i = \inf \pi_i A_i$. In particular, then, $\underline{x} = \underline{w}^0 \geq \underline{w}^1 \geq \ldots \geq \underline{w}^n$. We have $\underline{y} = \underline{w}^n$. We claim that $\underline{y} \in E_L^\Delta(A)$. For suppose that $\underline{z} \in A \cap L_Q$. Then $\underline{z} \leq \underline{y}$ and $\underline{z} \in A$. If $\underline{z} \neq \underline{y}$, then it differs in at least one compor. Suppose that $z_1 = y_1, \ldots, z_{i-1} = y_{i-1}, z_i < y_i$. Since $\underline{z} \leq \underline{y}$, we also have that $\underline{z} \leq \underline{w}^{i-1}$. But $z_i < y_i$ contradicts the definition of $\underline{w}^i$. Hence $\underline{z} = \underline{y}$.

As an immediate consequence we get the following theorem.

(2.6) **Theorem.** Let A be a closed upper set in the closed set Δ which is lower bounded. Then $A = \bigcup\limits_{\underline{y} \in E_L^\Delta(A)} U_Q(\underline{y}) \cap \Delta$ and

$$(2.7) \qquad I_A(\underline{x}) = \max\limits_{\underline{y} \in E_L^\Delta(A)} \min\limits_{1 \leq i \leq n} \alpha_{i,y_i}(\underline{x}) \quad \text{for } \underline{x} \in \Delta.$$

(2.8) **Remarks.** (i) Let A be as in Theorem 2.6 and suppose we also have that $I_A(\underline{x}) = \max\limits_{\underline{y} \in E} \min\limits_{1 \leq i \leq n} \alpha_{i,y_i}(\underline{x})$ for $\underline{x} \in \Delta$. Then it is easy to show that if E is a subset of Δ, $E \supset E_L^\Delta(A)$; (ii) If we allow E, however, to be a subset of $\mathbb{R}^n$, then it is not necessarily true that $E \supset E_L^\Delta(A)$ unless Δ is a product set.

If we combine the above results we get the following consequence. Let $\Phi: \Delta \to [0, \infty)$ be an MMS. Then

$$(2.9) \qquad \Phi(\underline{x}) = \int_0^\infty \phi(\underline{x}, t) dt$$

where $\phi(\underline{x}, t) = I_{U_t}(\underline{x})$ and $U_t = \{\underline{x} \epsilon \Delta : \Phi(\underline{x}) \geq t\}$. Now if Δ is closed and lower bounded and if Φ is upper semicontinuous on Δ, then U_t is a closed upper set in Δ which is lower bounded. Hence we can write

$$(2.10) \qquad \phi(\underline{x}, t) = \max_{\underline{y} \epsilon P_t} \min_{1 \leq i \leq n} \alpha_{i, y_i}(\underline{x}) \qquad \text{for } \underline{x} \epsilon \Delta$$

where $P_t = E_L^{\Delta}(U_t)$. We summarize below.

(2.11) <u>Theorem</u>. Let Δ be closed and lower bounded in $\mathbb{R}^n$ and let $\Phi : \Delta \rightarrow [0, \infty)$ be an upper semicontinuous MMS. Then (2.9) and (2.10) are valid.

We call (2.10) the min path representation for ϕ on Δ.

2.2 <u>Special cases</u>

Even though we have obtained a min path representation for Φ on Δ, a min cut representation is not immediate. The problem is that although the complement of U_t in Δ is a lower set in Δ, it is in general open and not closed in Δ even if we assume that Φ is continuous. The material in the previous section does not extend easily to open sets. One interesting case where the complement is closed occurs when Δ is discrete. We shall consider this case here. We also show how we can obtain useful bounds on the system performance by making use of the alternative representation (see Remark (2.2)).

Let $B \subset \Delta$ be a lower set in Δ. We say that $\underline{z} \epsilon \Delta$ is an upper extreme point of the lower set B relative to Δ if $U_Q(\underline{z}) \cap \overline{B} \cap \Delta = \{\underline{z}\}$. The set of all upper extreme points of B relative to Δ is denoted by $E_U^{\Delta}(B)$. The following results are proved in an analogous manner as (2.5) and (2.6).

(2.12) <u>Lemma</u>. Let Δ be a closed set in $\mathbb{R}^n$. Let B be a closed lower set in Δ which is upper bounded. Then for $\underline{x} \epsilon \Delta$, it follows that $\underline{x} \epsilon B$ if and only if $\underline{x} \leq \underline{z}$ for some $\underline{z} \epsilon E_U^{\Delta}(B)$.

(2.13) <u>Theorem.</u> Let B be as in Lemma 2.12. Then $B = \bigcup_{\underline{z} \in E_U^\Delta(B)} L_Q(\underline{z}) \cap \Delta$

and so

$$(2.14) \qquad I_B(\underline{x}) = \max_{\underline{z} \in E_U^\Delta(B)} \; \min_{1 \le i \le n} \beta_{i,z_i}(\underline{x}) \qquad \text{for } \underline{x} \in \Delta$$

where $\beta_{i,t}(\underline{x}) = 1$ if $x_i \le t$ and 0 otherwise.

(2.15) <u>Remark.</u> It should be noted that the β defined in this paper is not the same as the β in Block and Savits (1982).

Suppose now that $\Phi: \Delta \to [0,\infty)$ is an MMS. Recall the representation

$$\Phi(\underline{x}) = \int_0^\infty \phi(\underline{x},t)dt$$

where $\phi(\underline{x},t) = I_{U_t}(\underline{x})$ and $U_t = \{\underline{x}: \Phi(\underline{x}) \ge t\}$. We also have the alternative representation (see Remark 2.2)

$$\Phi(\underline{x}) = \int_0^\infty \xi(\underline{x},t)dt$$

where $\xi(\underline{x},t) = I_{U_t^{(o)}}(\underline{x})$ and $U_t^{(o)}(\underline{x}) = \{\underline{x}: \Phi(\underline{x}) > t\}$. Set $L_t = \{\underline{x}: \Phi(\underline{x}) \le t\}$ and $L_t^{(o)} = \{\underline{x}: \Phi(\underline{x}) < t\}$. Both L_t and $L_t^{(o)}$ are lower sets in Δ.

(2.16) <u>Theorem.</u> Suppose Δ is a finite set in $\mathbf{R}^n$ and $\phi: \Delta \to [0,\infty)$ is an MMS. We then have the dual representation

$$(2.17) \qquad \phi(\underline{x},t) = \max_{\underline{y} \in P_t} \; \min_{1 \le i \le n} \alpha_{i,y_i}(\underline{x}) = \min_{\underline{z} \in K_t^{(o)}} \; \max_{1 \le i \le n} \alpha_{i,z_i}^{(o)}(\underline{x})$$

where $P_t = E_L^\Delta(U_t)$, $K_t^{(o)} = E_U^\Delta(L_t^{(o)})$ and $\alpha_{i,t}^{(o)}(\underline{x}) = 1$ if and only if $x_i > t$.

<u>Proof.</u> We only need prove the second equality because of Theorem 2.11. Since Δ is finite, $L_t^{(o)}$ is a closed lower set in Δ which is upper bounded. Hence, by Theorem 2.13, we can write

$$I_{L_t^{(o)}}(\underline{x}) = \max_{\underline{z} \in E_U^\Delta(L_t^{(o)})} \min_{1 \leq i \leq n} \beta_{i,z_i}(\underline{x}).$$

Consequently,

$$\phi(\underline{x},t) = 1 - I_{L_t^{(o)}}(\underline{x}) = \min_{\underline{z} \in K_t^{(o)}} \max_{1 \leq i \leq n} (1 - \beta_{i,z_i}(\underline{x}))$$

$$= \min_{\underline{z} \in K_t^{(o)}} \max_{1 \leq i \leq n} \alpha_{i,z_i}^{(o)}(\underline{x}).$$

In the finite case, then, we have both a min path and a min cut representation (compare with Block and Savits (1982)).

In the general case, however, the set $L_t^{(o)}$ is not closed. However if we assume that Φ is continuous, the set L_t will be closed.

(2.18) **Theorem.** Let Φ be a continuous MMS on a compact set Δ. Then we have for $t \geq 0$ and $\underline{x} \in \Delta$

(2.19) $$\phi(\underline{x},t) = \max_{\underline{y} \in P_t} \min_{1 \leq i \leq n} \alpha_{i,y_i}(\underline{x}) \geq \xi(\underline{x},t) = \min_{\underline{z} \in K_t} \max_{1 \leq i \leq n} \alpha_{i,z_i}^{(o)}(\underline{x})$$

where $K_t = E_U^\Delta(L_t)$ and the other quantities are as in Theorem 2.16.

Proof. Since Φ is continuous and Δ is compact, the set L_t is a closed lower set which is upper bounded. Everything follows as in the proof of Theorem 2.16.

Although the representation (2.19) is not exactly what we desire, it does allow us to give bounds on the system performance function.

Let $\underline{X} = (X_1, \ldots, X_n)$ be a state vector, i.e., $\underline{X}: \Omega \to \Delta$ is a random vector. For each i, set $F_i(t) = P(X_i \leq t)$ and $\bar{F}_i(t) = 1 - F_i(t)$. Let Φ be a continuous MMS on Δ and assume Δ is compact so that (2.19) holds. We let $F(t) = P(\Phi(\underline{X}) \leq t)$ and $\bar{F}(t) = P(\Phi(\underline{X}) > t)$. We also let $\bar{F}(t-) = P(\Phi(\underline{X}) \geq t)$.

(2 20) <u>Theorem</u>. Let Φ be a continuous MMS on a compact Δ and let $t \geq 0$.

The following bounds always hold:

$$\sup_{\underline{y} \in P_t} P(\bigcap_{i=1}^{n} \{X_i \geq y_i\}) \leq \bar{F}(t-)$$

and

$$\bar{F}(t) \leq \inf_{\underline{z} \in K_t} P(\bigcup_{i=1}^{n} \{X_i > z_i\}).$$

In order to obtain further bounds, we need some preliminary definitions.

(2.21) <u>Definition</u>. Let $\Delta < \mathbb{R}^n$. We say that $\underline{x} \in \Delta$ is <u>biregular</u> if for every $\delta > 0$, there exists $\underline{u}, \underline{v} \in \Delta$ such that $(1-\delta)\underline{x} < \underline{u} < \underline{x} < \underline{v} < (1+\delta)\underline{x}$. A measure μ on Δ is said to be <u>biregular</u> if we can write $\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2$ where Δ_0 is countable, $\mu(\Delta_1) = 0$ and every $\underline{x} \in \Delta_2$ is biregular. A random vector $\underline{X}: \Omega \to \Delta$ is called <u>biregular</u> if the induced measure $\mu = P o \underline{X}^{-1}$ is biregular.

(2.22) <u>Remark</u>. Clearly every random vector $\underline{X}$ on a discrete compact set Δ is biregular.

(2.23) <u>Definition</u>. Let $\{a_t : t \in T\}$ be a collection of real numbers. We define

$$\prod_{t \in T} a_t = \inf_{\substack{S \subset T \\ S \text{ finite}}} \prod_{t \in S} a_t$$

$$\coprod_{t \in T} a_t = \sup_{\substack{S \subset T \\ S \text{ finite}}} \coprod_{t \in S} a_t.$$

(2.24) <u>Theorem</u>. Let Φ be a continuous MMS on a compact $\Delta \subset \mathbb{R}^n$ and suppose that $\underline{X}$ is biregular. Then for all but a countable set of t, we have the following results:

(a) If the X_i are associated,

$$\prod_{\underline{z} \in K_t} P(\bigcup_{i=1}^{n} \{X_i > z_i\}) \leq \bar{F}(t) = \bar{F}(t-) \leq \coprod_{\underline{y} \in P_t} P(\bigcap_{i=1}^{n} \{X_i \geq y_i\}).$$

(b) If the X_i are independent,

$$\prod_{\underline{z} \in K_t} \coprod_{i=1}^{n} \bar{F}_i(z_i) \leq \bar{F}(t) = \bar{F}(t-) \leq \coprod_{\underline{y} \in P_t} \prod_{i=1}^{n} \bar{F}(y_{\bar{i}}).$$

<u>Proof.</u> Since (b) easily follows from (a), we only prove (a). Now for every

$t \geq 0$, we have $U_t = \{\underline{x} \in \Delta: \Phi(\underline{x}) \geq t\} \supset \{\underline{x} \in \Delta: \Phi(x) > t\} = U_t^{(o)}$. Since $\Sigma_t =$

$U_t \setminus U_t^{(o)} = \{\underline{x} \in \Delta: \Phi(\underline{x}) = t\}$, it follows that $P(\underline{X} \in U_t) = P(\underline{X} \in U_t^{(o)})$ except for

possibly countably many t. We now only consider such t. But $U_t^{(o)} = \bigcup_{m=1}^{\infty} U_{t+\frac{1}{m}}$

and so given $\epsilon > 0$ there exists an m such that $P(\underline{X} \in U_t^{(o)}) \leq P(\underline{X} \in U_{t+\frac{1}{m}}) + \epsilon$. We

set $s = t + \frac{1}{m}$. Note that U_s is a compact upper set and $U_t = \bigcup_{\underline{y} \in P_t} U_Q(\underline{y}) \cap \Delta$ where

$P_t = E_L^{\Delta}(U_t)$. We now use the assumption that $\underline{X}$ is biregular. Thus

$\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2$ with Δ_0 countable, $P(\underline{X} \in \Delta_1) = 0$ and every $\underline{x} \in \Delta_2$ is biregular.

We choose a finite subset $\Delta_0^* \subset \Delta_0$ and a compact subset $\Delta_2^* \subset \Delta_2$ such that

$P(\underline{X} \in \Delta_0 - \Delta_0^*) < \epsilon$ and $P(\underline{X} \in \Delta_2 - \Delta_2^*) < \epsilon$. We thus have

$$\bar{F}(t-) = P(\underline{X} \in U_t) = P(\underline{X} \in U_t^{(o)}) \leq P(\underline{X} \in U_s) + \epsilon$$

$$\leq P(\underline{X} \in U_s \cap \Delta_0^*) + P(\underline{X} \in U_s \cap \Delta_2^*) = 3\epsilon.$$

Suppose that $\underline{x} \in U_s \cap \Delta_2^*$. Then $\Phi(\underline{x}) \geq s > t$. By continuity of Φ and the fact that

$\underline{x}$ is biregular we can find $\underline{u} \in \Delta$ such that $\underline{u} < \underline{x}$ and $\Phi(\underline{u}) > t$; i.e., $\underline{u} \in U_t$.

Hence there exists $\underline{y} \in P_t$ with $\underline{y} < \underline{u}$ which implies that $\underline{x} \in U_Q^o(\underline{y}) \cap \Delta$. Thus

$\{U_Q^o(\underline{y}) \cap \Delta: \underline{y} \in P_t\}$ is an open cover of the compact set $U_s \cap \Delta_2^*$. Let

$U_Q^o(\underline{y}^1), \ldots, U_Q^o(\underline{y}^m)$ be a finite subcover: $U_s \cap \Delta_2^* \subset \bigcup_{j=1}^{m} U_Q(\underline{y}^j) \cap \Delta$. Also since

Δ_0^* is finite, there exist finitely many $\underline{y}^{m+1}, \ldots, \underline{y}^p \in P_t$ such that

$U_s \cap \Delta_0^* \subset \bigcup_{j=m+1}^{p} U_Q(\underline{y}^j) \cap \Delta$. We may thus write $\bar{F}(t-) \leq P(\underline{X} \in \bigcup_{j=1}^{p} U_Q(\underline{y}^j)) + 3\epsilon$.

Since the X_i's are associated we obtain

$$\bar{F}(t-) \le \coprod_{j=1}^{p} P(\underline{X} \in U_Q(\underline{y}^j)) + 3\epsilon = \coprod_{j=1}^{p} P(\bigcap_{i=1}^{n} \{X_i \ge y_i^j\}) + 3\epsilon$$

$$\le \coprod_{\underline{y} \in P_t} P(\bigcap_{i=1}^{n} \{X_i \ge y_i\}) + 3\epsilon.$$

Since ϵ was arbitrary, we are done.

A similar argument works for the lower bound.

3. Applications of the Decomposition

3.1 The Barlow-Wu structure function

As in Block and Savits (1982) we assume there is a binary coherent system with min path sets $P_1,\ldots,P_p$ and min cut sets $K_1,\ldots,K_k$. Then we define

$$(3.1) \qquad \zeta(\underline{x}) = \max_{1 \le r \le p} \min_{i \in P_r} x_i = \min_{1 \le s \le k} \max_{i \in K_s} x_i$$

where $0 \le x_i$ for $i = 1,\ldots,n$. Here x_i need not be integer valued as in Block and Savits (1982). The function $\zeta(\underline{x})$ is the analog of the structure function considered by Barlow and Wu (1978).

First it is clear that $\zeta(t\,\underline{x}) = t\,\zeta(\underline{x})$ for all $t \ge 0$. Thus as in Section 1

$$\zeta(\underline{x}) = \int_0^\infty \zeta(\underline{x},t)dt$$

where $\zeta(\underline{x},t) = I_{[0,\zeta(\underline{x})]}(t) = I_{U_t}(\underline{x})$ and $U_t = \{\underline{x}: \zeta(\underline{x}) \ge t\}$. Now it is easy to see that $U_t = tU_1$ and so $\zeta(\underline{x},t) = \zeta(t^{-1}\underline{x},1)$ for all $t > 0$. Also

$$E_L(U_t) = \{\underline{z}: L_Q(\underline{z}) \cap U_t = \{\underline{z}\}\} = \{\underline{z}: \zeta(\underline{z}) \ge t \text{ and for } \underline{x} \lneq \underline{z}, \zeta(\underline{x}) < t\}$$

$$= t\{\underline{z}: \zeta(\underline{z}) \ge 1 \text{ and for } \underline{x} \lneq \underline{z}, \zeta(\underline{x}) < 1\} = t\,E_L(U_1).$$

(3.2) **Theorem.** $\zeta(\underline{x}) = \int_0^\infty \zeta(\underline{x},t)dt$ where

$$\zeta(\underline{x},t) = I_{U_t}(\underline{x}) = \max_{\underline{y} \in E_L(U_t)} \min_{1 \le i \le n} \alpha_{i,y_i}(\underline{x})$$

$$= \zeta(\alpha_t(\underline{x}))$$

where $\alpha_t(\underline{x}) = (\alpha_{1,t}(\underline{x}),\ldots,\alpha_{n,t}(\underline{x}))$.

Proof. The first and second identities follow from Theorem 1.3 and 2.6. To establish the last equality we have by the arguments immediately preceding the theorem that

$$\zeta(\underline{x},t) = \zeta(t^{-1}\underline{x},1) = \max_{\underline{y} \in E_L(U_1)} \min_{1 \le i \le n} \alpha_{i,y_i}(t^{-1}\underline{x}).$$

Now $\{\underline{y}: \underline{y} \in E_L(U_1)\} = \{\text{min path vectors corresponding to } P_1,\ldots,P_r\}$ as is easily demonstrated. Thus

$$\zeta(\underline{x},t) = \max_{1 \le r \le p} \min_{i \in P_r} \alpha_{i,1}(t^{-1}\underline{x})$$

$$= \max_{1 \le r \le p} \min_{i \in P_r} \alpha_{i,t}(\underline{x}) = \zeta(\alpha_t(\underline{x})).$$

(3.3) Remarks. i) The above result is the generalization of Lemma 6.3 and Theorem 6.4 of Block and Savits (1982). The connection is clear if we notice that $\zeta_k(\alpha(\underline{z}))$ of that paper is $\zeta(\underline{x},k)$.

ii) A similar result is possible in terms of the min max representation if we have (3.1) for $0 \le x_i \le M_i$, $i = 1,\ldots,n$ by using (2.19).

3.2 Representation for homogeneous functions

In the finite state case discrete functions of the type (3.1) have been characterized in Block and Savits (1982). In the continuous case a similar but more general result can be obtained for homogeneous functions.

(3.4) <u>Theorem</u>. Assume $\Phi:[0,\infty)^n \to [0,\infty)$ is a right continuous nondecreasing function such that $\Phi(t\underline{x}) = t\Phi(\underline{x})$ for all $t>0$. Then

$$\Phi(\underline{x}) = \max_{\underline{y}\in E_L(U_1)} \min_{y_i\neq 0} y_i^{-1}x_i$$

where $E_L(U_1) = \{\underline{y}: \phi(\underline{y}) \geq 1 \text{ and for } \underline{x} \lneqq \underline{y}, \Phi(\underline{x}) < 1\}$.

<u>Proof</u>. From Theorem 1.3 we have

$$\Phi(\underline{x}) = \int_0^\infty \phi(\underline{x},t)dt$$

where $\phi(\underline{x},t) = I_{U_t}(\underline{x})$ and $U_t = \{\underline{x}: \Phi(\underline{x}) \geq t\}$. Then by Remark (1.9i) we have for $t > 0$

$$\phi(\underline{x},t) = \max_{\underline{y}\in E_L(U_t)} \min_{y_i\neq 0} \alpha_{i,y_i}(\underline{x}).$$

By arguments similar to those preceding Theorem 4.1 we have for $t > 0$

$$\phi(\underline{x},t) = \max_{\underline{y}\in E_L(U_1)} \min_{y_i\neq 0} \alpha_{i,y_i}(t^{-1}\underline{x}).$$

By a straightforward argument it follows that

$$\phi(\underline{x},t) = I_{[0,\ \max_{\underline{y}\in E_L(U_1)} \min_{y_i\neq 0} y_i^{-1}x_i]}(t)$$

and so

$$\Phi(\underline{x}) = \int_0^\infty \phi(\underline{x},t)dt = \max_{\underline{y}\in E_L(U_1)} \min_{y_i\neq 0} y_i^{-1}x_i.$$

(3.5) <u>Remarks</u>. i) It is easy to check that the condition of Theorem 3.4 is equivalent to the condition $U_t = t U_1$ for all $t > 0$.

ii) A similar representation is possible using the dual representation of Section 2.2 under the appropriate assumptions. In particular from (2.19) for $\underline{x} \in \Delta = \{\underline{x}: 0 \leq x_i \leq M, \quad i = 1,\ldots,n\}$,

$$\Phi(\underline{x}) = \int_0^\infty \xi(\underline{x},t)dt$$

where

$$\xi(\underline{x},t) = \min_{\underline{z} \in E_U^{\Delta}(L_t)} \max_{1 \le i \le n} \alpha_{i,z_i}^{(o)}(\underline{x}) \quad \text{for } \underline{x} \in \Delta.$$

and it follows similarly that

$$\Phi(\underline{x}) = \min_{\underline{z} \in E_U^{\Delta}(L_1)} \max_{1 \le i \le n} z_i^{-1} x_i \quad \text{for } \underline{x} \in \Delta$$

where $z_i^{-1} x_i = \begin{cases} 0 & \text{if } x_i = o \\ \infty & \text{if } x_i > 0 \text{ and } z_i = 0. \end{cases}$

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Multistate structure functions on state spaces which are not necessarily finite are discussed. Integral representations are derived which extend the decomposition of Block and Savits (1982). Bounds for the structure function are obtained and the representation is applied to the Barlow-Wu structure function and to nondecreasing homogeneous functions.

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